

On products of groups with abelian subgroups of small index

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Abstract

It is proved that every group of the form $G = AB$ with two subgroups A and B each of which is either abelian or has a quasicyclic subgroup of index 2 is soluble of derived length at most 3. In particular, if A is abelian and B is a locally quaternion group, this gives a positive answer to Question 18.95 of "Kourovka notebook" posed by A.I. Sozutov.

1 Introduction

Let the group $G = AB$ be the product of two subgroups A and B , i.e. $G = \{ab \mid a \in A, b \in B\}$. It was proved by N. Itô that the group G is metabelian if the subgroups A and B are abelian (see [1, Theorem 2.1.1]).

In connection with Itô's theorem a natural question is whether every group $G = AB$ with abelian-by-finite subgroups A and B is metabelian-by-finite [1, Question 3] or at least soluble-by-finite. However, this seemingly simple question is very difficult to attack and only partial results in this direction are known. A positive answer was given for linear groups G by the second author in [7] (see also [8]) and for residually finite groups G by J. Wilson [1, Theorem 2.3.4]. Furthermore, N.S. Chernikov proved that every group $G = AB$ with central-by-finite subgroups A and B is soluble-by-finite (see [1, Theorem 2.2.5]).

It is natural to consider first groups $G = AB$ where the two factors A and B have abelian subgroups with small index. There are a few known results in the case when both factors A and B have an abelian subgroup of index at most 2. It was shown in [3] that G is soluble and metacyclic-by-finite if A and B have cyclic subgroups of index at most 2, and it is proved in [2] that G is soluble if A and B are periodic locally dihedral subgroups. A more general result that $G = AB$ is soluble if each of the factors A and B is either abelian or generalized dihedral was obtained in [4] by another approach. Here a group is called generalized dihedral if it contains an abelian subgroup of index 2 and an involution which inverts the elements of this subgroup. Clearly dihedral groups and locally dihedral groups, i.e. groups with a local system of dihedral subgroups, are generalized dihedral.

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We recall that a group is called quasicyclic (or a Prüfer group) if it is an infinite locally cyclic p -group for some prime p . It is well-known that quasicyclic subgroups of abelian groups are their direct factors. Furthermore, it seems to be known and will be shown below that every non-abelian group having a quasicyclic subgroup of index 2 is either an infinite locally dihedral or a locally quaternion group. It should be noted that for each prime p , up to isomorphism, there exists a unique locally dihedral group whose quasicyclic subgroup is a p -group, and there is only one locally quaternion group. These and other details about such groups can be found in [6], p. 45 - 50.

Theorem 1.1. *Let the group $G = AB$ be the product of two subgroups A and B each of which is either abelian or has a quasicyclic subgroup of index 2. Then G is soluble with derived length at most 3. Moreover, if the subgroup B is non-abelian and X is its quasicyclic subgroup, then $AX = XA$ is a metabelian subgroup of index 2 in G .*

As a direct consequence of this theorem, we have an affirmative answer to Question 18.95 of the "Kourovka notebook" posed by A.I. Sozutov.

Corollary 1.2. *If a group $G = AB$ is the product of an abelian subgroup A and a locally quaternion subgroup B , then G is soluble.*

It is also easy to see that if each of the factors A and B in Theorem 1.1 has a quasicyclic subgroup of index 2, then their quasicyclic subgroups are permutable. As a result of this the following holds.

Corollary 1.3. *Let the group $G = A_1A_2 \cdots A_n$ be the product of pairwise permutable subgroups A_1, \dots, A_n each of which contains a quasicyclic subgroup of index 2. Then the derived subgroup G' is a direct product of the quasicyclic subgroups and the factor group G/G' is elementary abelian of order 2^m for some positive integer $m \leq n$.*

The notation is standard. If H is a subgroup of a group G and $g \in G$, then the normal closure of H in G is the normal subgroup of G generated by all conjugates of H in G , and g^G is the conjugacy class of G containing g , respectively.

2 Preliminary lemmas

Our first lemma lists some simple facts concerning groups with quasicyclic subgroups of index 2 which will be used without further explanation.

Lemma 2.1. *Let G be a non-abelian group containing a quasicyclic p -subgroup X of index 2 and $y \in G \setminus X$. Then $y^2 \in X$ and the following statements hold:*

- 1) every subgroup of X is characteristic in G ;
- 2) the group G is either locally dihedral or locally quaternion;
- 3) the derived subgroup G' coincides with X ;
- 4) every proper normal subgroup of G is contained in X ;

- 5) if G is locally quaternion, then $p = 2$, $y^4 = 1$, $x^y = x^{-1}$ for all $x \in X$, the center $Z(G)$ coincides with $\langle y^2 \rangle$ and is contained in every non-trivial subgroup of G , the coset yX coincides with the conjugacy class $y^G = y^X$;
- 6) if G is locally dihedral, then $y^2 = 1$, $x^y = x^{-1}$ for all $x \in X$, $Z(G) = 1$ and the coset yX coincides with the conjugacy class $y^G = y^X$ for $p > 2$ and $Z(G)$ is the subgroup of order 2 in X for $p = 2$;
- 7) the factor group $G/Z(G)$ is locally dihedral.

Proof. In fact, only statement 2) needs an explanation. Clearly $G = X\langle y \rangle$ for some $y \in G$ with $y^2 \in X$ and each cyclic subgroup $\langle x \rangle$ of X is normal in G . Therefore for $p > 2$ we have $y^2 = 1$ and either $x^y = x$ or $x^y = x^{-1}$. Since X contains a unique cyclic subgroup of order p^n for each $n \geq 1$, the equality $x^y = x$ for some $x \neq 1$ holds for all $x \in X$, contrary to the hypothesis that G is non-abelian. Therefore $x^y = x^{-1}$ for all $x \in X$ and hence the group G is locally dihedral. In the case $p = 2$ each subgroup $\langle x \rangle$ of X properly containing the subgroup $\langle y^2 \rangle$ has index 2 in the subgroup $\langle x, y \rangle$. If x is of order 2^n for some $n > 3$, then the element y can be chosen such that either $y^4 = 1$ and $\langle x, y \rangle$ is a generalized quaternion group with $x^y = x^{-1}$ or $y^2 = 1$ and $\langle x, y \rangle$ is one of the following groups: dihedral with $x^y = x^{-1}$, semidihedral with $x^y = x^{-1+2^{n-2}}$ or a group with $x^y = x^{1+2^{n-2}}$ (see [5], Theorem 5.4.3). It is easy to see that from this list only generalized quaternion and dihedral subgroups can form an infinite ascending series of subgroups, so that the 2-group G can be either locally quaternion or locally dihedral, as claimed. \square

Lemma 2.2. *Let G be a group and M an abelian minimal normal p -subgroup of G for some prime p . Then the factor group $G/C_G(M)$ has no non-trivial finite normal p -subgroup.*

Proof. Indeed, if $N/C_G(M)$ is a finite normal p -subgroup of $G/C_G(M)$ and x is an element of order p in M , then the subgroup $\langle N, x \rangle$ is finite and so the centralizer $C_M(N)$ is a non-trivial normal subgroup of G properly contained in M , contrary to the minimality of M . \square

We will say that a subset S of G is normal in G if $S^g = S$ for each $g \in G$ which means that $s^g \in S$ for every $s \in S$.

Lemma 2.3. *Let G be a group and let A, B be subgroups of G . If a normal subset S of G is contained in the set AB and $S^{-1} = S$, then the normal subgroup of G generated by S is also contained in AB . In particular, if i is an involution with $i^G \subseteq AB$ and N is the normal closure of the subgroup $\langle i \rangle$ in G , then $AN \cap BN = A_1B_1$ with $A_1 = A \cap BN$ and $B_1 = AN \cap B$.*

Proof. Indeed, if $s, t \in S$, then $t = ab$ and $(s^{-1})^a = cd$ for some elements $a, c \in A$ and $b, d \in B$. Therefore $s^{-1}t = s^{-1}ab = a(s^{-1})^ab = (ac)(db) \in AB$ and hence the subgroup $\langle s \mid s \in S \rangle$ is contained in AB and normal in G . Moreover, if N is a normal subgroup of G and $N \subseteq AB$, then it is easy to see that $AN \cap BN = (AN \cap B)N = (AN \cap B)N = (A \cap BN)(AN \cap B)$ (for details see [1], Lemma 1.1.3) \square

The following slight generalization of Itô's theorem was proved in [7] (see also [8], Lemma 9).

Lemma 2.4. *Let G be a group and let A, B be abelian subgroups of G . If H is a subgroup of G contained in the set AB , then H is metabelian.*

3 The product of an abelian group and a group containing a quasicyclic subgroup of index 2

In this section we consider groups of the form $G = AB$ with an abelian subgroup A and a subgroup $B = X \langle y \rangle$ in which X is a quasicyclic subgroup of index 2 and $y \in B \setminus X$.

Lemma 3.1. *Let the group $G = AB$ be the product of an abelian subgroup A and a non-abelian subgroup B with a quasicyclic subgroup X of index 2. If G has non-trivial abelian normal subgroups, then one of these is contained in the set AX .*

Proof. Suppose the contrary and let \mathcal{N} be the set of all non-trivial normal subgroups of G contained in the derived subgroup G' . Then $A_G = 1$ and $ANX \neq AX$ for each $N \in \mathcal{N}$. Since $G = AB = AX \cup AXy$ and $AX \cap AXy = \emptyset$, for every $N \in \mathcal{N}$ the intersection $NX \cap AXy$ is non-empty and so $G = ANX$. Moreover, as $X = B' \leq G'$ by Lemma 2.1, it follows that $G' = DNX$ with $D = A \cap G'$. It is also clear that $G = \langle A, X \rangle$, because otherwise $\langle A, X \rangle = AX$ is a normal subgroup of index 2 in G . In particular, $A \cap X = 1$.

For each $N \in \mathcal{N}$ we put $A_N = A \cap BN$ and $B_N = AN \cap B$. Then $A_N N = B_N N = A_N B_N$ by [1], Lemma 1.1.4, and the subgroup B_N is not contained in X , because otherwise N is contained in the set AX , contrary to the assumption. Let $X_N = B_N \cap X$ and $C_N = A_N \cap NX_N$. Then X_N is a subgroup of index 2 in B_N and $C_N N = NX_N$ is a normal subgroup of G , because $(NX_N)^X = NX_N$ and $(NC_N)^A = NC_N$. Put $M = \bigcap_{N \in \mathcal{N}} N$.

Since $G = ANX$ for each $N \in \mathcal{N}$, the factor group G/N is metabelian by Lemma 2.4. Therefore also the factor group G/M is metabelian and so its derived subgroup G'/M is abelian. Clearly if $M = 1$, then $D = A \cap G' \leq A_G = 1$ and so $G' = \bigcap_{N \in \mathcal{N}} NX = X$, contrary to the assumption. Thus M is an abelian minimal normal subgroup of G . If M is finite, then its centralizer $C_G(M)$ in G contains X and so the group $G = A(MX)$ is metabelian. Then $G' = DMX$ is abelian and hence $D = A \cap G' \leq A_G = 1$. Therefore $G' = MX$ and so X is a normal subgroup of G , contrary to the assumption. Thus M is infinite and then $MX_M = C_M M$ is an abelian normal subgroup of G with finite cyclic subgroup X_M whose order is a prime power $p^k \geq 1$.

If M contains no elements of order p , then $M \cap C_M = 1$ and $C_M^{p^k} = 1$, so that C_M is of order p^k . But then the subgroups A_M and B_M are of order $2p^k$ and hence the subgroup $A_M M = B_M M = A_M B_M$ is finite, contrary to the choice of M . Thus M is an elementary abelian p -subgroup and so the factor group $\bar{G} = G/C_G(M)$ has no non-trivial finite normal p -subgroups by Lemma 2.2. In particular, the center of \bar{G} has no elements of order p . On the other hand, $G = AMX$ and $G' = DMX$ with $D = A \cap G'$, so that $\bar{G} = \bar{A}\bar{X}$ is a metabelian group and its derived subgroup $\bar{G}' = \bar{D}\bar{X}$ is abelian. Thus $\bar{D} \leq \bar{A} \cap \bar{G}'$ is a central subgroup of \bar{G} which contains no elements of

order p . But then \bar{X} and so each of its subgroups are normal in \bar{G} . This final contradiction completes the proof. \square

It should be noted that if in Lemma 3.1 the subgroup B is locally dihedral, then the group $G = AB$ is soluble by [4], Theorem 1.1. Therefore the following assertion is an easy consequence of this lemma.

Corollary 3.2. *If the group $G = AB$ is the product of an abelian subgroup A and a locally dihedral subgroup B containing a quasicyclic subgroup X of index 2, then $AX = XA$ is a metabelian subgroup of index 2 in G .*

Proof. Indeed, let H be a maximal normal subgroup of G with respect to the condition $H \subseteq AX$. If $X \leq H$, then $AH = AX$ is a metabelian subgroup of index 2 in G by Itô's theorem. In the other case the intersection $H \cap X$ is finite and hence HX/H is the quasicyclic subgroup of index 2 in BH/H . Since $G/H = (AH/H)(BH/H)$ is the product of the abelian subgroup AH/H and the locally dihedral subgroup BH/H , the set $(AH/H)(HX/H)$ contains a non-trivial normal subgroup F/H of G/H by Lemma 3.1. But then F is a normal subgroup of G which is contained in the set AX and properly contains H . This contradiction completes the proof. \square

In the following lemma $G = AB$ is a group with an abelian subgroup A and a locally quaternion subgroup $B = X \langle y \rangle$ in which X is the quasicyclic 2-subgroup of index 2 and y is an element of order 4, so that $x^y = x^{-1}$ for each $x \in X$ and $z = y^2$ is the unique involution of B . It turns out that in this case the conjugacy class z^G of z in G is contained in the set AX .

Lemma 3.3. *If $G = AB$ and $A \cap B = 1$, then the intersection $z^A \cap AXy$ is empty.*

Proof. Suppose the contrary and let $z^a = bxy$ for some elements $a, b \in A$ and $x \in X$. Then $b^{-1}z = (xy)^{a^{-1}}$ and from the equality $(xy)^2 = z$ it follows that $(b^{-1}z)^4 = 1$ and $b^{-1}zb^{-1}z = z^{a^{-1}}$. Therefore $b^{-1}z^ab^{-1} = zz^a$ and hence $bz^ab = z^az$. As $z^a = bxy$, we have $b(bxy)b = (bxy)z$ and so $bxyb = xyz$. Thus $(xy)^{-1}b(xy) = zb^{-1}$. Furthermore, $bxyb^{-1} = (zb^{-1})^a$, so that $bzb^{-1} = ((zb^{-1})^2)^a = (xy)^{-a}b^2(xy)^a$, i.e. the elements z and b^2 are conjugate in G by the element $g = b^{-1}(xy)^{-a}$. Since $g = cd$ for some $c \in B$ and $d \in A$, we have $b^2 = z^g = z^d$ and so $z = (b^2)^{d^{-1}} = b^2$, contrary to the hypothesis of the lemma. Thus $z^A \cap AXy = \emptyset$, as desired. \square

Theorem 3.4. *Let the group $G = AB$ be the product of an abelian subgroup A and a locally quaternion subgroup B . If X is the quasicyclic subgroup of B , then $AX = XA$ is a metabelian subgroup of index 2 in G . In particular, G is soluble of derived length at most 3.*

Proof. Let Z be the center of B , N the normal closure of Z in G and $X = B'$, so that X is the quasicyclic subgroup of index 2 in B . If $A \cap B \neq 1$, then Z is contained in $A \cap B$ by statement 4) of Lemma 2.1 and so $N = Z$. Otherwise it follows from Lemma 2.3 that $N = Z^G = Z^A$ is contained in the set AX . Then N is a metabelian normal subgroup of G by Lemma 2.4 and the factor group BN/N is locally dihedral by statement 7) of Lemma 2.1. Since the factor group $G/N = (AN/N)(BN/N)$ is the product of an abelian

subgroup AN/N and the locally dihedral subgroup BN/N , it is soluble by [4], Theorem 1.1, and so the group G is soluble.

Now if $X \leq N$, then $AN = AX$ is a metabelian subgroup of index 2 in G and so the derived length of G does not exceed 3. In the other case the intersection $N \cap X$ is finite and hence NX/N is the quasicyclic subgroup of index 2 in BN/N . Therefore $AX = XA$ by Corollary 3.2 and this completes the proof. \square

4 The product of a locally quaternion and a generalized dihedral subgroup

In this section we consider groups of the form $G = AB$ with a locally quaternion subgroup A and a generalized dihedral subgroup B . The main part is devoted to the proof that every group G of this form has a non-trivial abelian normal subgroup. In what follows $G = AB$ is a group in which $A = Q\langle c \rangle$ with a quasicyclic 2-subgroup Q of index 2 and an element c of order 4 such that $a^c = a^{-1}$ for each $a \in Q$ and $B = X \rtimes \langle y \rangle$ with an abelian subgroup X and an involution y such that $x^y = x^{-1}$ for each $x \in X$.

Let $d = c^2$ denote the involution of A . The following assertion is concerned with the structure of the centralizer $C_G(d)$ of d in G . It follows from statement 4) of Lemma 2.1 that the normalizer of every non-trivial normal subgroup of A is contained in $C_G(d)$.

Lemma 4.1. *The centralizer $C_G(d)$ is soluble.*

Proof. Indeed, if $Z = \langle d \rangle$, then the factor group $C_G(d)/Z = (A/Z)(C_B(d)Z/Z)$ is a product of the generalized dihedral subgroup A/Z and the subgroup $C_B(d)Z/Z$ which is either abelian or generalized dihedral. Therefore $C_G(d)/Z$ and thus $C_G(d)$ is a soluble group by [4], Theorem 1.1, as claimed. \square

The following lemma shows that if G has no non-trivial abelian normal subgroup, then the index of A in $C_G(d)$ does not exceed 2.

Lemma 4.2. *If $C_B(d) \neq 1$, then either $C_X(d) = 1$ or G contains a non-trivial abelian normal subgroup.*

Proof. If $X_1 = C_X(d)$, then X_1 is a normal subgroup of B and $C_G(d) = AC_B(d)$. Therefore the normal closure $N = X_1^G$ is contained in $C_G(d)$, because $X_1^G = X_1^{BA} = X_1^A$. Since $C_G(d)$ and so N is a soluble subgroup by Lemma 4.1, this completes the proof. \square

Consider now the normalizers in A of non-trivial normal subgroups of B .

Lemma 4.3. *Let G have no non-trivial abelian normal subgroup. If U is a non-trivial normal subgroup of B , then $N_A(U) = 1$. In particular, $A \cap B = 1$.*

Proof. Indeed, if $N_A(U) \neq 1$, then $d \in N_A(U)$ and so the normal closure $\langle d \rangle^G = \langle d \rangle^B$ is contained in the normalizer $N_G(U) = N_A(U)B$. Since $N_A(U) \neq A$, the subgroup $N_A(U)$ is either finite or quasicyclic, so that $N_G(U)$ and thus $\langle d \rangle^G$ is soluble. This contradiction completes the proof. \square

Lemma 4.4. *If $C_X(d) = 1$, then G contains a non-trivial abelian normal subgroup.*

Proof. Since $G = AB$, for each $x \in B$ there exist elements $a \in A$ and $b \in B$ such that $d^x = ab$. If $b \notin X$, then $b = a^{-1}d^x$ is an element of order 2 and so $d^x ad^x = a^{-1}$. As $a^{2^k} = d$ for some $k \geq 0$, it follows that $d^x dd^x = d$ and hence $ab = d^x = (d^x)^d = (ab)^d = ab^d$. Therefore $b^d = b$ and so $b \in C_B(d)$. In particular, if $C_B(d) = 1$, then $b \in X$, so that in this case the conjugacy class $d^G = d^B$ is contained in the set AX .

Assume that $C_B(d) \neq 1$ and the group G has no non-trivial normal subgroup. Then $C_X(d) = 1$ by Lemma 4.2 and without loss of generality $C_B(d) = \langle y \rangle$. Then $G = (A\langle y \rangle)X$ and so the quasicyclic subgroup Q of A is normalized by y . In particular, $d^y = d$ and the subgroup $Q\langle y \rangle$ can be either abelian or locally dihedral. We consider first the case when y centralizes Q and show that in this case the conjugacy class d^G is also contained in the set AX .

Indeed, otherwise there exist elements $a \in A$ and $b, x \in B$ such that $d^x = ab$ and $b \notin X$. Then $b \in C_B(d) = \langle y \rangle$ by what was proved above, so that $b = y$ and $d^x = ay$. As $d^B = d^{\langle y \rangle X} = d^X$, we may suppose that $x \in X$. But then $d^{x^{-1}} = (d^x)^y = ay = d^x$ and hence $d^{x^2} = d$. Therefore $x^2 \in \langle y \rangle$ and so $x^2 = 1$. In particular, if X has no involution, then $d^G = d^X \subseteq AX$. We show next that the case with an involution $x \in X$ cannot appear.

Clearly in this case x is a central involution in B and so the subgroup $D = \langle d, x \rangle$ generated by the involutions d and x is dihedral. It is easy to see that d and x cannot be conjugate in G and the center of D is trivial, because otherwise the centralizer $C_G(x)$ properly contains B , contradicting Lemma 4.3. Thus dx is an element of infinite order and so $D = \langle dx \rangle \rtimes \langle x \rangle$ has no automorphism of finite order more than 2. On the other hand, if $u \in A$ and $v \in B$, then $D^{uv} = D$ if and only if $D^u = D$ and $D^v = D$, so that $N_G(D) = N_A(D)N_B(D)$. Therefore $N_A(D) = \langle d \rangle$ and hence $z = (dx)^2$ is an element of infinite order in $N_B(D)$. But then $z \in X$ and so $\langle z \rangle$ is a normal subgroup of B normalized by d , again contradicting Lemma 4.3. Thus X has no involution, as claimed.

Finally, if N is the normal closure of the subgroup $\langle d \rangle$ in G , then $AN = NX = A_1X_1$ with $A_1 = A \cap NX$ and $X_1 = AN \cap X$ by Lemma 2.3. Therefore the subgroup A_1X_1 is soluble by Theorem 3.4, so that N and hence G has a non-trivial abelian normal subgroup, contrary to our assumption.

Thus the subgroup $Q\langle y \rangle$ is locally dihedral and so y inverts the elements of Q . Since $A = Q\langle c \rangle$ with $a^c = a^{-1}$ for all $a \in A$, the element cy centralizes Q and hence the subgroup $Q\langle cy \rangle$ is abelian. But then the group $G = (Q\langle cy \rangle)B$ as the product of an abelian and a generalized dihedral subgroup is soluble by [4], Theorem 1.1. This final contradiction completes the proof. \square

Theorem 4.5. *Let the group $G = AB$ be the product of a locally quaternion subgroup A and a generalized dihedral subgroup B . Then G is soluble. Moreover, if B has a quasicyclic subgroup of index 2, then G is metabelian.*

Proof. If $A \cap X \neq 1$, then the centralizer $C_G(d)$ is of index at most 2 in G and so G is soluble by Lemma 4.1. Let N be a normal subgroup of G maximal with respect to the condition $A \cap NX = 1$. Then $BN = (A \cap BN)B$ and the subgroup $A \cap BN$ is of order at most 2. Therefore the subgroup N

is soluble and the factor group $G/N = (AN/N)(BN/N)$ is the product of the locally quaternion subgroup AN/N and the subgroup BN/N which is either abelian or generalized dihedral. Hence it follows from Theorem 3.4 and Lemmas 4.2 and 4.4 that G/N has a non-trivial abelian normal subgroup M/N . Put $L = MQ \cap MX$, $Q_1 = Q \cap MX$ and $X_1 = MQ \cap X$. Then $L = MQ_1 = MX_1$ and $Q_1 \neq 1$, because $A \cap MX \neq 1$ by the choice of M . It is also clear that L is a soluble normal subgroup of G , because $(MQ_1)^A = MQ_1$ and $(MX_1)^B = MX_1$. Therefore the factor group G/L and so the group G is soluble if AL/L is of order 2. In the other case AL/L is locally dihedral and BL/L is abelian or generalized dihedral. Since $G/L = (AL/L)(BL/L)$, it follows that G/L and so G is soluble by [4], Theorem 1.1. Moreover, if the subgroup X is quasicyclic, then the subgroups Q and X centralize each other by [1], Corollary 3.2.8, so that QX is an abelian normal subgroup of index 4 in G and thus G is metabelian. \square

The *Proof of Theorem 1.1* is completed by a direct application of Corollary 3.2, Theorem 3.4 and Theorem 4.5. \square

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